Misc. Cosmology

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1 Coordinates

t proper time
τ conformal time
z redshift
a scale factor
r proper distance
x comoving distance

1.1 redshift-scale factor-time relation

$$1 + z = \frac{1}{a}$$

$$\frac{da}{dt} = aH = \frac{H_0 E(z)}{1 + z}$$

$a \propto t^{1/2}$, radiation domination
$a \propto t^{2/3}$, matter domination
$a \propto e^t$, Λ domination

1.2 proper-comoving relation

$$r = ax$$

comoving distance

$$\frac{dx}{dt} = -c(1 + z)$$

$$dx = c\frac{da}{a^2 H}$$

$$x = c\int_a^1 \frac{da'}{a'^2 H} = c\int_0^z \frac{dz'}{H}$$

1.3 line relation

$$\lambda(z) = \lambda_0(1+z) = \frac{\lambda_0}{a}$$
$$\nu(z) = \frac{\nu_0}{(1+z)} = \nu_0 a$$
$$v(z) = cz$$

The above is not the actual recession velocity. It is an ok approximation for low redshift, but it's just a proxy for the redshift of the line. The recession velocity is technically

$$\begin{aligned} v_{\rm rec}(z) &= \frac{dr}{dt} = \frac{da}{dt}x + a\frac{dx}{dt} \\ &= \frac{da}{dt}x = caH\int_a^1 \frac{da'}{a'^2 H} \\ &= c\frac{H}{1+z}\int_0^z \frac{dz'}{H} \end{aligned} \tag{1}$$

2 Friedmann equations

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{2}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{4}$$

3 Cosmological EOS

$$w = \frac{p}{\rho}, w_{\rm b} = 1/3, w_{\rm cdm} = 0, w_\Lambda = -1 \tag{5}$$

$$\rho \propto a^{-3(1+w)} \tag{6}$$

$$H^2 = H_0^2 E(z)^2 = H_0^2\left[\Omega_{\rm r}(1+z)^4 + \Omega_{\rm m}(1+z)^3 + \Omega_\Lambda + \Omega_k(1+z)^2\right] \tag{7}$$

4 Photon-baryon fluid sound speed

$$\frac{\partial T}{T} = 3\frac{\partial\rho_{\rm b}}{\rho_{\rm b}} = 4\frac{\partial\rho_\gamma}{\rho_\gamma} \to \frac{\partial\rho_{\rm b}}{\partial\rho_\gamma} = \frac{4}{3}\frac{\rho_{\rm b}}{\rho_\gamma}$$

$$v_s^2 = \left(\frac{\partial p}{\partial\rho}\right)_S = \frac{\partial p_\gamma}{\partial(\rho_\gamma + \rho_{\rm b})} = \frac{1}{3}\frac{\partial\rho_\gamma}{\partial\rho_\gamma}\frac{1}{(1+\partial\rho_{\rm b}/\partial\rho_\gamma)^{-1}} = \frac{1}{3+4\rho_{\rm b}/\rho_\gamma} \sim \frac{1}{3}$$

5 Scale-invariant power spectrum

Scale invariance means that the power spectrum of the potential is

$$\Delta_\Phi^2(k) = \text{const}$$

$$P_\Phi \propto \Phi^2 \propto k^{-3}$$

$$\nabla^2\Phi \propto \delta \rightarrow k^2\Phi \propto \delta$$

$$P_\delta \propto \delta^2 \propto (k^2\Phi)^2 \propto k^4 k^{-3} \propto k$$

6 Power spectrum evolution

$$P_{\text{prim}}(k) = Ak_s^n$$

$$P(k,z) = T(k)^2 G(z)^2 P_{\text{prim}}(k)$$

Turns over at $k_{\text{eq}} \sim 0.01$ h/Mpc.