

FFTs and power spectra cheatsheet

Casey W. Stark

1 Fourier transforms and FFTs

The most common Fourier convention in cosmology is

$$\begin{aligned}\hat{f}(\mathbf{k}) &= \int d^3x f(\mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x}) \\ f(\mathbf{x}) &= \frac{1}{(2\pi)^3} \int d^3k \hat{f}(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x})\end{aligned}\tag{1}$$

Here x is the real space coordinate, k is the spectral space coordinate, and $f(x)$ is a dimensionless field, which makes $\hat{f}(k)$ a volume.

The Fast Fourier Transform (FFT) is the discrete version of this. For 1D, this is

$$\begin{aligned}b_l &= \sum_{x=0}^{N-1} a_x \exp(-i2\pi xl/N) \\ a_x &= \frac{1}{N} \sum_{l=0}^{N-1} b_l \exp(i2\pi xl/N)\end{aligned}\tag{2}$$

and for 3D,

$$\begin{aligned}b_{lmn} &= \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \sum_{z=0}^{N-1} a_{xyz} \exp(-i2\pi xl/N) \exp(-i2\pi ym/N) \exp(-i2\pi zn/N) \\ a_{xyz} &= \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} b_{lmn} \exp(i2\pi xl/N) \exp(i2\pi ym/N) \exp(i2\pi zn/N)\end{aligned}\tag{3}$$

There are many FFT implementations, but we almost always use FFTW. Note that the FFTW uses the `FFTW_FORWARD` and `FFTW_BACKWARD` flags to indicate the direction of the transform. `FFTW_FORWARD` means there is a $-$ in the exp, which luckily agrees with the convention above, but can change from paper to paper. This does not matter for many common operations, but it's something to be aware of if the order and sign of the coefficients matters (like in FFT Poisson solves).

2 The correlation function and power spectrum

The correlation of functions $f(x)$ and $g(x)$ is

$$\begin{aligned}\xi(\mathbf{x}) &= \langle f(\mathbf{x}')g(\mathbf{x}' + \mathbf{x}) \rangle \\ &= \frac{\int d^3x' f(\mathbf{x}')g(\mathbf{x}' + \mathbf{x})}{\int d^3x'}\end{aligned}\quad (4)$$

The angle bracket operator is the volume-weighted average, as written out above. Note that the correlation is always defined over a periodic volume $\int d^3x = V$.

The power spectrum is defined as the Fourier transform of the correlation.

$$P(\mathbf{k}) = \int d^3x (f(\mathbf{x}')g(\mathbf{x}' + \mathbf{x})) \exp(-i\mathbf{k} \cdot \mathbf{x}) \quad (5)$$

The correlation function is relatively expensive to compute directly since it is $O(N^2)$. Instead, we usually compute the power spectrum from the Fourier coefficients since FFTs are only $O(N \log N)$.

$$\begin{aligned}P(\mathbf{k}) &= \frac{1}{\int d^3x} \int_{x'} \int_x f(\mathbf{x}')g(\mathbf{x}' + \mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x}) d^3x' d^3x \\ &= \frac{1}{V(2\pi)^6} \int_{x'} \int_x \left[\int_{k'} \hat{f}(\mathbf{k}') \exp(i\mathbf{k}' \cdot \mathbf{x}') d^3k' \right] \left[\int_{k''} \hat{g}(\mathbf{k}'') \exp(i\mathbf{k}'' \cdot (\mathbf{x}' + \mathbf{x})) d^3k'' \right] \exp(-i\mathbf{k} \cdot \mathbf{x}) d^3x' d^3x \\ &= \frac{1}{V(2\pi)^6} \int_{k'} \int_{k''} \hat{f}(\mathbf{k}') \hat{g}(\mathbf{k}'') \left[\int_x \int_{x'} \exp(-i\mathbf{x} \cdot (\mathbf{k} - \mathbf{k}')) \exp(-i\mathbf{x}' \cdot (-\mathbf{k}'' - \mathbf{k}')) d^3k' d^3k'' \right] d^3x' d^3x \\ &= \frac{1}{V(2\pi)^6} \int_{k'} \int_{k''} \hat{f}(\mathbf{k}') g(\mathbf{k}'') V^2 \delta_D(\mathbf{k} - \mathbf{k}') \delta_D(\mathbf{k}' + \mathbf{k}'') d^3k' d^3k'' \\ &= \frac{1}{V(2\pi)^6} V^2 V_k^2 \hat{f}(\mathbf{k}) \hat{g}(-\mathbf{k})\end{aligned}$$

$$P(\mathbf{k}) = \frac{\hat{f}(\mathbf{k}) \hat{g}^*(\mathbf{k})}{V} \quad (6)$$

This is the power spectrum as a field. If the field is isotropic, the 1d power spectrum should contain all the information of the 3d version, so we average this field over k shells.

$$P(k)(2\pi)^3 \delta^D(\mathbf{k} - \mathbf{k}') = \langle f(\mathbf{k}) g^*(\mathbf{k}') \rangle \quad (7)$$

For the discrete version, we choose some binning in k , where k_i is the low bin edge for bin i .

$$\begin{aligned}
P_{lmn} &= \frac{\hat{f}_{lmn}\hat{g}_{lmn}^*}{V} \\
P_i &= \frac{\sum_j P_j}{\sum_j 1}, \text{ where } k_i \leq k_j < k_{i+1} \\
\bar{k}_i &= \frac{\sum_j P_j k_j}{\sum_j P_j}
\end{aligned} \tag{8}$$